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ON THE DISSIPATIVITY OF UNCONTROLLABLE SYSTEMS

M.K. Çamlıbel
Dogus University
81010 Acibadem
Istanbul Turkey
Tilburg University
P.O. Box 90153
5000 LE Tilburg The Netherlands
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven The Netherlands
e-mail: K.Camlibel@uvt.nl

Jan C. Willems
University of Leuven
Kasteelpark Arenberg 10
B-3001 Leuven-Heverlee Belgium
Jan.Willems@esat.kuleuven.ac.be
www.esat.kuleuven.ac.be/∼jwillems

Madhu N. Belur
University of Groningen
P.O. Box 800
9700 AV Groningen The Netherlands
M.N.Belur@math.rug.nl
www.math.rug.nl/∼madhu

***Abstract*— This paper deals with dissipativity of uncontrollable linear time-invariant systems with quadratic supply rates and storage functions. A definition of dissipativity appropriate for this class of systems is given. We present a necessary and sufficient condition for dissipativeness in the single input / single output case.**

Keywords: **Behaviors, dissipativity, storage functions, controllability, observability.**

I. INTRODUCTION

The roots of the theory of dissipative systems can be found in the early papers on electrical circuit theory. In [2], the notion of positive realness was introduced in the context of circuit synthesis. It was shown in this classic paper that a rational function g is the driving point impedance of a circuit consisting of a finite number of positive resistors, inductors, capacitors, and transformers if and only if g is positive real. Starting in the late fifties and early sixties, positive realness came to play a key role also in systems and control theory, through what we now call positive real (or KYP) lemma. In [5] dissipativity was conceptualized in terms of the storage function and the supply rate.

One of the assumptions in the study of dissipativity has almost always been controllability of the system. The very definition of a dissipative system is often given for the controllable case. There is, however, no reason to make the definition of an 'energy'-related concept like dissipativity dependent on a concept like controllability not related to energy at all. Even if there is a certain relationship between these two concepts, it should follow from the definitions instead of being imposed.

This paper is a step towards a dissipativity theory for uncontrollable systems in a behavioral context, for linear time-invariant systems with quadratic supply rates and storage functions. After giving a definition of dissipativity, we present a necessary and sufficient condition for a (not necessarily controllable) single-input single-output behavior to be dissipative.

II. PRELIMINARIES

A (linear time-invariant differential) behavior (for a detailed treatment of behavioral theory, we refer to [4]) is the set of solutions, assumed infinitely differentiable for ease of exposition, of a system of linear constant coefficient differential equations

$$R(\frac{d}{dt})w = 0$$

where $R \in \mathbb{R}^{\mathrm{q}\times\mathrm{w}}[\xi]$ is a polynomial matrix. Note that

$$\mathfrak{B} = \ker(R(\frac{d}{dt})). \tag{1}$$

The set of all such behaviors will be denoted by $\mathfrak{L}^{\mathrm{w}}$.

A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$ is said to be *controllable* if for any w_- and $w_+ \in \mathfrak{B}$ there exist a $T > 0$ and a $w \in \mathfrak{B}$ such that $w(t) = w_-(t)$ for all $t < 0$ and $w(t) = w_+(t - T)$ for all $t \geq T$. It is well-known that a behavior is controllable if and only if $\mathrm{rank}(R(\lambda))$ is the same for all $\lambda \in \mathbb{C}$.

On the other extreme of controllability are autonomous behaviors. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$ is *autonomous* if for w_1, $w_2 \in \mathfrak{B}$, $w_1(t) = w_2(t)$ for all $t < 0$ implies that $w_1 = w_2$.

A basic result of behavioral theory states that every behavior can be written as a direct sum of a (unique) controllable one and a (non-unique) autonomous one.

For obvious reasons, representations of the type (1) are called *kernel representations*. There are many other ways of representing a behavior. A representation of the type

$$R(\frac{d}{dt})w = M(\frac{d}{dt})\ell \tag{2}$$

with real R, M polynomial matrices is called a *latent variable representation*. In this case, $w \in \mathfrak{B}$ if there exists a *latent variable* trajectory ℓ such that this system of differential equations is satisfied. The variables w are called *manifest variables*. A particular case of this are *image representations*,

$$w = M(\frac{d}{dt})\ell,$$

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whence $\mathfrak{B} = \mathrm{im}(M(\frac{d}{dt}))$. It can be shown that a behavior is controllable if and only if it allows an image representation.

Let $\mathfrak{B}$ be a behavior with $w = (w_1, w_2)$. We call w_2 *observable* from w_1 if $(w_1, w_2'), (w_1, w_2'') \in \mathfrak{B}$ implies $w_2' = w_2''$. It turns out that this is the case if and only if there exists a polynomial matrix F such that $(w_1, w_2) \in \mathfrak{B}$ implies $w_2 = F(\frac{d}{dt})w_1$. Analogously, we call a latent variable representation *observable* if, whenever $(w, \ell'), (w, \ell'')$ satisfy (2), then $\ell' = \ell''$. It can be shown that a controllable $\mathfrak{B}$ admits an observable image representation.

Every behavior $\mathfrak{B}$ also admits an input-output representation. After a reordering, if need be, of the components of the manifest variable w, we obtain $w = \mathrm{col}(u, y)$ where u denote inputs and y outputs (see [4] for precise definitions of these concepts). An input/output partition of $\mathfrak{B}$ corresponds to a kernel of the form

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u, w = (u, y)$$

with the properties that P is square, $\det P \neq 0$, and $P^{-1}Q$ is a matrix of (proper) rational functions. We call this an *input-output representation* of $\mathfrak{B}$. The number of y components is given by $\mathrm{rank}(R)$.

Two-variable polynomial matrices can be used in the theory of dissipative systems in a very effective way. Let $\mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$ denote the set of real two-variable polynomial matrices in the indeterminates ζ and η. An element of this set, say Φ, is a finite sum

$$\Phi(\zeta, \eta) = \sum_{i,j} \Phi_{ij} \zeta^i \eta^j.$$

To each such Φ, we associate the *bilinear differential form*

$$L_\Phi(v, w) = \sum_{i,j} (\frac{d^i}{dt^i} v)^\top \Phi_{ij} (\frac{d^j}{dt^j} w).$$

Note that L_Φ is mapping from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q_2})$ to $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$. If Φ is square then it induces a *quadratic differential form* given by

$$Q_\Phi(w) = L_\Phi(w, w).$$

The QDF Q_Φ, or simply Φ, is said to be *nonnegative along* $\mathfrak{B}$ if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{B}$.

Bilinear and quadratic differential forms have been studied in detail in [6]. The (related) operators $^*, ^\bullet, \partial, ^\star$ play an important role in this. Define
$^* : \mathbb{R}^{q_1 \times q_2}[\zeta, \eta] \to \mathbb{R}^{q_2 \times q_1}[\zeta, \eta]$ by $\Phi^*(\zeta, \eta) = \Phi^\top(\eta, \zeta)$
$^\bullet : \mathbb{R}^{q_1 \times q_2}[\zeta, \eta] \to \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$ by $\overset{\bullet}{\Phi}(\zeta, \eta) = (\zeta + \eta)\Phi(\zeta, \eta)$,
$\partial : \mathbb{R}^{q_1 \times q_2}[\zeta, \eta] \to \mathbb{R}^{q_1 \times q_2}[\xi]$ by $\partial(\Phi)(\xi) = \Phi(-\xi, \xi)$,
and $^\star : \mathbb{R}^{q_1 \times q_2}[\xi] \to \mathbb{R}^{q_1 \times q_2}[\xi]$ by $F^\star(\xi) = F^\top(-\xi)$.
Note that $\partial(\Phi^*) = \partial(\Phi)^\star$, $L_{\overset{\bullet}{\Phi}} = \frac{d}{dt} L_\Phi$, and $\partial(^\bullet) = 0$. In fact, $\Phi \in \mathrm{im}(^\bullet)$ if and only if $\Phi \in \ker(\partial)$.

III. Dissipativity of uncontrollable systems

The behavior $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$ (not necessarily controllable) is said to be *dissipative* with respect to the storage function $Q_\Phi, \Phi \in \mathbb{R}^{\mathrm{w} \times \mathrm{w}}[\zeta, \eta]$ if there exists a latent variable representation $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$ of $\mathfrak{B}$ and a $\Psi \in \mathbb{R}^{\ell \times \ell}[\zeta, \eta]$ such that the *dissipation inequality*

$$\frac{d}{dt} Q_\Psi(\ell) \leq Q_\Phi(w)$$

holds for all (w, ℓ) that satisfy $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$. Q_Ψ is called the *storage function*. When the dissipation inequality holds as an equality, we say that $\mathfrak{B}$ is Φ*-lossless*. If the storage function acts on w, i.e. if $\Psi \in \mathbb{R}^{\mathrm{w} \times \mathrm{w}}[\zeta, \eta]$ and

$$\frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w) \tag{3}$$

for all $w \in \mathfrak{B}$, then we call the system dissipative with an *observable storage function*.

Non-negative storage functions are very important in applications, but we will not consider them in the present paper. Our storage functions need not be sign definite.

In the sequel, we confine attention to single-input single-output behaviors, and (mostly) to observable storage functions. We assume that the behavior (with the manifest variable $w = \mathrm{col}(u, y)$) is governed by

$$\gamma(\frac{d}{dt})\beta(\frac{d}{dt})y = \gamma(\frac{d}{dt})\alpha(\frac{d}{dt})u \tag{4}$$

where α, β, γ are scalar polynomials such that α and β are co-prime and γ is monic. For the supply rate, we take

$$\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

i.e. $Q_\Phi(w) = 2uy$. Since α and β are coprime, there exist polynomials p and q such that

$$\alpha p + \beta q = 1. \tag{5}$$

A decomposition of the behavior $\mathfrak{B}$, defined by (4), into controllable and autonomous parts can be obtained as follows. Let

$$R_c = \begin{bmatrix} \alpha & -\beta \end{bmatrix}$$

and also let

$$R_a^\nu = \begin{bmatrix} \gamma\alpha & -\gamma\beta \\ q + \alpha\nu & p - \beta\nu \end{bmatrix}$$

for some polynomial ν. Define

$$\mathfrak{B}_c = \ker R_c(\frac{d}{dt})$$

and

$$\mathfrak{B}_a^\nu = \ker R_a^\nu(\frac{d}{dt})$$

Now, $\mathfrak{B} = \mathfrak{B}_c \oplus \mathfrak{B}_a^\nu$ yields a desired decomposition. In fact, ν parametrizes all possible autonomous parts. An alternative representation for the autonomous part is given by

$$\mathfrak{B}_a^\nu = \{w \mid w = \begin{bmatrix} p'(\frac{d}{dt}) \\ q'(\frac{d}{dt}) \end{bmatrix} \ell \text{ and } \gamma(\frac{d}{dt})\ell = 0\}$$

where $p' := p - \beta\nu$ and $q' = -q - \alpha\nu$.

Dissipativity of a controllable behavior $\mathfrak{B}$ is a well-understood subject. The present paper is an attempt to investigate dissipativity of uncontrollable systems. We begin by recalling well-known results for the controllable case. Before that, we need some nomenclature. Let $\chi \in \mathbb{R}[\xi]$ be para-Hermitian, i.e. $\chi^\star = \chi$. A polynomial $\kappa \in \mathbb{R}[\xi]$ is called a *symmetric factor* of χ if $\chi = \kappa^\star\kappa$. It is easy to see that a symmetric factor exists if and only if $\chi(i\omega)$ is real (hence χ is even) and non-negative for $\omega \in \mathbb{R}$.

The following proposition gives an answer to the question when a controllable system is dissipative. The proof follows from propositions 5.2, 5.6, and theorem 6.4 of [6].

Proposition 1: Let $\mathfrak{B}$ be given by (4) and let $\gamma = 1$. The following statements are equivalent.

1) $\mathfrak{B}$ is Φ-dissipative with an observable storage function,
2) $\mathfrak{B}$ is Φ-dissipative,
3) $\alpha\beta^\star + \alpha^\star\beta$ admits a symmetric factor.

The main contribution of the present paper is the following theorem which provides a necessary and sufficient condition for the dissipativity of an uncontrollable behavior.

Theorem 2: *Let $\mathfrak{B}$ be given by* (4)*. Assume that γ has no roots on the imaginary axis. Then, the following statements are equivalent.*

1) *$\mathfrak{B}$ is Φ-dissipative with an observable storage function.*
2) *$\alpha\beta^\star + \alpha^\star\beta$ admits a symmetric factor that is coprime with γ.*

Proof: ($1 \Rightarrow 2$): (by contradiction) Suppose that 1 holds but 2 does not hold. In other words, suppose that $\mathfrak{B}$ is Φ-dissipative and all symmetric factors of $\alpha\beta^\star + \alpha^\star\beta$ have a common root with γ. As $\mathfrak{B}$ is Φ-dissipative, its controllable part is also dissipative, i.e., there exists a Ψ such that

$$\begin{aligned}\Phi(\zeta,\eta) &- (\zeta+\eta)\Psi(\zeta,\eta)\\ &= F^\top(\zeta)F(\eta) + R_c^\top(\zeta)G(\zeta,\eta) + G^*(\zeta,\eta)R_c(\eta)\end{aligned} \tag{6}$$

for some polynomial matrices $F \in \mathbb{R}^{1\times 2}[\xi]$ and $G \in \mathbb{R}^{1\times 2}[\zeta,\eta]$. For a trajectory w_c such that $F(\frac{d}{dt})w_c = R_c(\frac{d}{dt})w_c = 0$, $Q_{\Phi-\overset{\bullet}{\Psi}}(w_c) = 0$. Consequently, dissipativity of $\mathfrak{B}$ implies that the bilinear differential form $L_{\Phi-\overset{\bullet}{\Psi}}(w, w_c)$ must vanish for all $w \in \mathfrak{B}$ and for all w_c such that $F(\frac{d}{dt})w_c = R_c(\frac{d}{dt})w_c = 0$. Therefore, there exist $J, K \in \mathbb{R}^{2\times 2}[\zeta,\eta]$ such that

$$\begin{aligned}\Phi(\zeta,\eta) &- (\zeta+\eta)\Psi(\zeta,\eta)\\ &= R^\top(\zeta)J(\zeta,\eta) + K(\zeta,\eta)\begin{bmatrix}F(\eta)\\ R_c(\eta)\end{bmatrix}.\end{aligned}$$

Pre-multiplication by $\begin{bmatrix}p(\zeta) & -q(\zeta)\end{bmatrix}$, post-multiplication by $\mathrm{col}(\beta(\eta),\alpha(\eta))$, and evaluation at $\zeta = -\xi$ and $\eta = \xi$ result in

$$p^\star\alpha - q^\star\beta = \gamma^\star\tilde{j} + \tilde{k}\delta. \tag{7}$$

for some polynomials $\tilde{j}$ and $\tilde{k}$ where $\delta = F\mathrm{col}(\beta,\alpha)$. The above polynomial equation is satisfied if and only if for $\lambda \in \mathbb{C}$, the implication

$$\delta(\lambda) = \gamma(-\lambda) = 0 \Rightarrow p(-\lambda)\alpha(\lambda) = q(-\lambda)\beta(\lambda)$$

holds. On the other hand, the polynomial δ is nothing but a symmetric factor of $\alpha\beta^\star + \alpha^\star\beta$. To see this, postmultiply (6) by $\mathrm{col}(\beta(\eta),\alpha(\eta))$, premultiply by the transpose of $\mathrm{col}(\beta(\eta),\alpha(\eta))$, and evaluate at $\zeta = -\xi$ and $\eta = \xi$. So $\delta(\lambda) = 0$ implies that $\alpha(\lambda)\beta(-\lambda) + \alpha(-\lambda)\beta(\lambda) = 0$. Together with $p(-\lambda)\alpha(\lambda) - q(-\lambda)\beta(\lambda) = 0$, this implies that

$$\begin{bmatrix}\beta(-\lambda) & \alpha(-\lambda)\\ p(-\lambda) & -q(-\lambda)\end{bmatrix}\begin{bmatrix}\alpha(\lambda)\\ \beta(\lambda)\end{bmatrix} = 0.$$

However, the first factor on the left hand side is nonsingular due to (5) and the second factor is nonzero as α and β are coprime. This means that if the polynomial equation (7) has a solution then δ and $\gamma^\star$ are necessarily coprime. Define $\delta' := \delta^\star$. Note that δ' is a symmetric factor of $\alpha\beta^\star + \alpha^\star\beta$ and (δ',γ) is coprime. We reach a contradiction as we found a symmetric factor of $\alpha\beta^\star + \alpha^\star\beta$ which is coprime with γ.

($2 \Rightarrow 1$): Let δ be a symmetric factor of $\alpha\beta^\star + \alpha^\star\beta$ which is coprime with γ. Define

$$\begin{aligned}U &:= \begin{bmatrix}\beta & p\\ \alpha & -q\end{bmatrix},\\ F &:= \begin{bmatrix}\delta^\star & 0\end{bmatrix}U^{-1},\\ G(\zeta,\eta) &:= \begin{bmatrix}p(\zeta)\alpha(\eta) - q(\zeta)\beta(\eta) & q(\zeta)p(\eta)\end{bmatrix}U^{-1}(\eta).\end{aligned}$$

Note that U is unimodular due to (5). Straightforward computation yields

$$F^\star F + R_c^\star\partial G + \partial G^* R_c = \partial\Phi.$$

Indeed, one can check above equation by pre-multiplying both sides by $U^\star$ and post-multiplying by U. Therefore, there exists an Ω such that

$$\begin{aligned}\Phi(\zeta,\eta) &- (\zeta+\eta)\Omega(\zeta,\eta)\\ &= F^\top(\zeta)F(\eta) + R_c^\top(\zeta)G(\zeta,\eta) + G^*(\zeta,\eta)R_c(\eta),\end{aligned}$$

and hence

$$Q_{\Phi-\overset{\bullet}{\Omega}}(w_c) \geq 0 \tag{8}$$

for all $w_c \in \mathfrak{B}_c$. Let $\mathfrak{B}_a$ be the autonomous part corresponding to $\nu = 0$. We claim that $\Psi = \Omega + \Theta + \mu\Gamma$ satisfies, for some $\mu \in \mathbb{R}$,

$$Q_{\Phi-\overset{\bullet}{\Psi}}(w) \geq 0 \tag{9}$$

for all $w \in \mathfrak{B}$ if Θ and Γ satisfy

i) $Q_\Theta(w_c) = 0$ for all $w_c \in \mathfrak{B}_c$,
ii) $w_c \in \mathfrak{B}_c$ and $Q_{\Phi-\overset{\bullet}{\Omega}}(w_c) = 0$ implies $L_{\overset{\bullet}{\Theta}}(w, w_c) = L_{\Phi-\overset{\bullet}{\Omega}}(w, w_c)$ for all $w \in \mathfrak{B}$,
iii) $Q_{\overset{\bullet}{\Gamma}}(w_c) = 0$ for all $w_c \in \mathfrak{B}_c$,
iv) $L_{\overset{\bullet}{\Gamma}}(\cdot, w_c) = 0$ for all $w_c \in \mathfrak{B}_c$, and
v) $Q_{\overset{\bullet}{\Gamma}}(w_a) < 0$ for all $0 \neq w_a \in \mathfrak{B}_a$.

To prove this, take any $w = w_a + w_c$ where $w_a \in \mathfrak{B}_a$ and $w_c \in \mathfrak{B}_c$.Then, we get

$$\begin{aligned} Q_{\Phi-\overset{\bullet}{\Psi}}(w) &= Q_{\Phi-\overset{\bullet}{\Psi}}(w_a) + 2L_{\Phi-\overset{\bullet}{\Psi}}(w_a, w_c) + Q_{\Phi-\overset{\bullet}{\Psi}}(w_c) \\ &= Q_{\Phi-\overset{\bullet}{\Psi}}(w_a) + 2L_{\Phi-\overset{\bullet}{\Omega}-\overset{\bullet}{\Theta}}(w_a, w_c) + Q_{\Phi-\overset{\bullet}{\Omega}}(w_c). \end{aligned}$$

Note that the first summand of the last line can be made arbitrarily large by choosing μ sufficiently small due to (v). The last summand is already nonnegative due to (8). Together with (ii), these imply that (9) holds for all $w \in \mathfrak{B}$ if we choose μ sufficiently small. To finish the proof, we will show the existence of Θ and Γ such that (i)-(v) are satisfied. To do so, take

$$\Gamma(\zeta, \eta) = \begin{bmatrix} \alpha(\zeta) \\ -\beta(\zeta) \end{bmatrix} X^\top(\zeta) L X(\eta) \begin{bmatrix} \alpha(\eta) & -\beta(\eta) \end{bmatrix} \quad (10)$$

where X induces a state map for $\mathfrak{B}_a$, the corresponding state model is given by

$$x = X(\frac{d}{dt})w_a, \quad \frac{d}{dt}x = Ax, \quad w_a = Cx,$$

for some matrices A and C with appropriate sizes, and L is such that $A^\top L + LA < 0$. Such an L exists since γ has no roots on the imaginary axis. It can be easily checked that (iii)-(v) are satisfied by the choice of (10). Therefore, it remains to show the existence of a Θ satisfying (i)-(ii). We know from [6, proposition 3.2 and proposition 3.5] that there exists Θ satisfying (i)-(ii) if and only if there exist two-variable polynomial matrices H, J, and K such that

$$\Theta(\zeta, \eta) = R_c^\top(\zeta) H(\zeta, \eta) + H^*(\zeta, \eta) R_c(\eta), \quad (11)$$

$$(\zeta + \eta)\Theta = R_c^\top(\zeta) G + R_a^\top(\zeta) J + K \begin{bmatrix} F(\eta) \\ R_c(\eta) \end{bmatrix}. \quad (12)$$

Consider the following partitions

$$H(\zeta, \eta)U(\eta) = \begin{bmatrix} H_1(\zeta, \eta) & H_2(\zeta, \eta) \end{bmatrix},$$
$$J(\zeta, \eta)U(\eta) = \begin{bmatrix} J_{11}(\zeta, \eta) & J_{12}(\zeta, \eta) \\ J_{21}(\zeta, \eta) & J_{22}(\zeta, \eta) \end{bmatrix},$$
$$U^\top(\zeta)K(\zeta, \eta) = \begin{bmatrix} K_{11}(\zeta, \eta) & K_{12}(\zeta, \eta) \\ K_{21}(\zeta, \eta) & K_{22}(\zeta, \eta) \end{bmatrix}.$$

Pre-multiplying (11)-(12) by $U^\top(\zeta)$, post-multiplying by $U(\eta)$, and eliminating $\Theta(\zeta, \eta)$, yields the system of polynomial equations:

$$J_{21}(\zeta, \eta) + K_{11}(\zeta, \eta)\delta(-\eta) = 0, \quad (13)$$

$$K_{12}(\zeta, \eta) + J_{22}(\zeta, \eta) = (\zeta + \eta)H_1^*(\zeta, \eta), \quad (14)$$

$$\begin{aligned} K_{21}(\zeta, \eta)\delta(-\eta) + \gamma(\zeta)J_{11}(\zeta, \eta) + G_1(\zeta, \eta) \\ = (\zeta + \eta)H_1(\zeta, \eta), \end{aligned} \quad (15)$$

$$\begin{aligned} K_{22}(\zeta, \eta) + \gamma(\zeta)J_{12}(\zeta, \eta) + G_2 \\ = (\zeta + \eta)(H_2(\zeta, \eta) + H_2^*(\zeta, \eta)) \end{aligned} \quad (16)$$

Note that (13) and (16) are solvable as soon as (14) and (15) are. Also note that (14) and (15) are solvable if and only if

$$\begin{aligned} K_{21}(\zeta, \eta)F_1(\eta) + \gamma(\zeta)J_{11}(\zeta, \eta) + G_1(\zeta, \eta) \\ = K_{12}^*(\zeta, \eta) + J_{22}^*(\zeta, \eta) \end{aligned}$$
$$\partial K_{21}\delta^\star + \gamma^\star \partial J_{11} + \partial G_1 = 0$$

are solvable. Clearly, the former equation is solvable as soon as the latter is. As δ and γ are coprime, the latter always admits a solution.

This ends the proof of theorem 2.

Remarks:

1. Note that the roots of $\alpha\beta^\star + \alpha^\star\beta$ are symmetric with respect to imaginary axis. Let $\gamma = \gamma_1\gamma_2$ where γ_1 has no symmetric roots with respect to imaginary axis and $\gamma_2 = \pm\gamma_2^\star$. Then, there exists a symmetric factor of $\alpha\beta^\star + \alpha^\star\beta$ which is coprime with γ only if $\alpha\beta^\star + \alpha^\star\beta$ and γ_2 are coprime. In particular, when γ has no symmetric roots with respect to imaginary axis (i.e., it has no even factor), the behavior $\mathfrak{B}$ is Φ-dissipative if and only if its controllable part is.

2. In [3] a sufficient condition for the passivity of uncontrollable multiple input / multiple output state space systems is given. For the single input / single output, the condition given in [3] comes down to the requirement that $\beta\gamma$ should have no symmetric roots with respect to the imaginary axis. The first remark shows that this is a special case of theorem 2.

3. When the controllable part is lossless, it can be shown that $\alpha\beta^\star + \alpha^\star\beta$ is identically zero. Thus, the coprimeness condition of theorem 2 holds only if γ is a constant and hence the behavior $\mathfrak{B}$ is controllable. On the other extreme, when $\alpha\beta^\star + \alpha^\star\beta$ is a constant, the controllable part is strictly dissipative, i.e. the dissipation inequality (3) holds with the strict inequality for all nonzero trajectories of $\mathfrak{B}_c$. Then, coprimeness condition of theorem 2 readily holds independently on the autonomous part.

IV. UNOBSERVABLE STORAGE FUNCTIONS

Theorem 2 deals with observable storage functions, i.e. storage functions that are only functions of the manifest variables. The use of uncontrollable systems and/or unobservable storage functions is of considerable importance. We discuss this in the present section.

Example: Consider the system

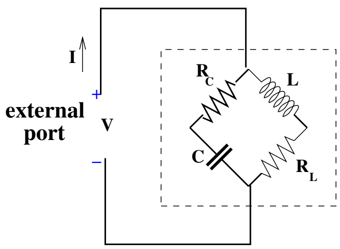
$$\frac{d}{dt}y = \frac{d}{dt}u.$$

It follows from theorem 2 that with respect to the supply rate uy, this system does not have an observable storage function. Consider however the latent variable representation given by

$$\begin{aligned} \frac{d}{dt}x_1 &= x_2 \\ \frac{d}{dt}x_2 &= 0 \\ y &= x_2 + u. \end{aligned}$$

If we allow the storage function to be a function of the latent variables, then this system becomes dissipative. To see this, consider the storage function $Q_\Psi(x) = kx_1x_2$. So, $Q_{\overset{\bullet}{\Psi}}(x) = k(\frac{d}{dt}x_1)x_2 + kx_1(\frac{d}{dt}x_2) = kx_2^2$. Then, $Q_\Phi(w) - Q_{\overset{\bullet}{\Psi}}(x) = uy - kx_2^2 = u^2 + ux_2 - kx_2^2$. It is easy to verify that this expression is nonnegative for all u and x_2 if $k \leq -1/4$.

An important area of application of the ideas of this article is the area of electrical circuits.



Consider for example the circuit shown, and regard (V, I) as manifest and the internal branch currents and voltages as latent variables (see [4], pages 10-13, 160-161, and 175-176 for a derivation of the equations and the analysis of the controllability and observability properties of this circuit). Of course, this circuit is dissipative with respect to the supply rate VI with the internal energy, $\frac{1}{2}CV_C^2 + \frac{1}{2}LI_L^2$ (in the obvious notation) as the storage function. If $CR_C \neq \frac{L}{R_L}$, this circuit is controllable and observable (in the sense that the branch currents and voltages are observable from the port voltage and current). However, when $CR_C = \frac{L}{R_L}$, the differential equation that governs (V, I) is

$$(\frac{R_C}{R_L} + CR_C\frac{d}{dt})V = (1 + CR_C\frac{d}{dt})R_CI.$$

The variables (V_C, I_L) are then unobservable from (V, I), and hence, the stored energy becomes an unobservable storage function. When $CR_C = \frac{L}{R_L}$ and $R_C \neq R_L$, then the manifest behavior is controllable. So, there exists an observable storage function. In fact, classical results from electrical circuit synthesis allow to conclude that the port behavior can also be realized using passive elements (resistors and one capacitor, in fact), and with observable branch currents and voltages. However, when $CR_C = \frac{L}{R_L}$ and $R_C = R_L$, the port behavior becomes uncontrollable and theorem 2 shows that there does not exist an observable storage function. In fact, in this case, it can be shown that there does not exist a passive synthesis with only one reactive element. So, in this sense, the realization which we started from is a minimal one. Of course, all this shows the limited relevance of the classical notion of minimal (controllable and observable) state space representations in the context of physical systems.

The most classical result of circuit theory is undoubtedly the fact that g is the driving point impedance of a circuit containing a finite number of passive resistors, capacitors, inductors, and transformers if and only if g is rational and positive real. This result was obtained by Brune [2] in his MIT Ph.D. dissertation. In 1949, Bott and Duffin [1] proved that transformers are not needed.

It seems to us that a more 'complete' version of this classical problem is to ask for the realization of a differential behavior. This problem is somewhat more general than the driving point impedance problem, because of the existence of uncontrollable systems. For example, a unit resistor realizes the transfer function of the system

$$\frac{d}{dt}V + V = \frac{d}{dt}I + I,$$

as its driving point impedance, but not its behavior (which admits, for example, the short circuit response $I(t) = e^{-t}, V(t) = 0$, not realized by the resistor). An example of a circuit that does realize this behavior exactly is the above circuit, with $L = 1, C = 1, R_L = 1, R_C = 1$, so this uncontrollable behavior *is* realizable.

This leads to two nice open problems:

Problem 1: *What behaviors $\mathfrak{B} \in \mathfrak{L}^2$ are realizable as the port behavior of a circuit containing a finite number of passive resistors, capacitors, inductors, and transformers?* It is easy to see that $\mathfrak{B}$ must be single input / single output, and that the transfer function must be rational and positive real. In addition $\mathfrak{B}$ must be passive, but in general with a non-observable storage function, and therefore it is not clear what this says in terms of $\mathfrak{B}$.

Problem 2: *Is it possible to realize a controllable single input / single output system with a rational positive real transfer function as the behavior of a circuit containing a finite number of passive resistors, capacitors, and inductors, but no transformers?* Note that in a sense this is the Bott-Duffin problem, the issue being that the Bott-Duffin synthesis procedure usually realizes a non-controllable system that has the correct transfer function (i.e., the correct controllable part), but not the correct behavior. There are standard synthesis procedures known that do realize the correct behavior, but they need transformers.

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