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# MODEL REDUCTION BY BALANCING FROM A BEHAVIORAL POINT OF VIEW

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## 1 Introduction

The concept of balancing consists in representing a system in state space form, so that the external properties of the system are reflected at the level of state. A balanced basis for the state space allows to determine which of the state variables have smaller contribution to the external behavior of the system. The problem of balancing has been given wide attention in the past, cfr. [1], [2], [3], [5] (where the notion of Riccati balancing was introduced), [6] (the pioneering work in this area, where Lyapunov balancing was discussed), [10] (of which the present work constitutes a generalization).

The interest in this topic has two main motivations. First, once a balanced state space representation has been computed, a heuristic procedure for model reduction consists in truncating the state vector, originally of $n$ components, to the first $k$ components. In the classical approach to balancing, as advocated in [6], the rationale behind this procedure is that states which require large energy to be reached and from which small energy future outputs can be extracted can in good conscience be neglected, if one wishes to obtain a model which approximately has the same impulse response as the original one. Moreover, the reduced order model obtained in this way enjoys some desirable properties, see [7]. A second reason why balanced state space realizations have been given such attention in the past is their use in concrete algorithms to perform Hankel-norm approximation, as illustrated in the seminal work [4].

In this note we consider the problem of model reduction by balancing from the behavioral point of view, as illustrated in [11, 12]. We consider general quadratic measures on the external signals of a system; Lyapunov and Riccati balancing are special cases of this situation. Using the concept of state map, introduced in [8], we describe how to compute a balanced state space representation from the original equations of the system, without the need to construct an intermediate state space representation. Once a balanced state space representation has been computed, a reduced order model of order $k$ is obtained by considering only the subsystem corresponding to the first $k$ components of the state.

Why should one choose the behavioral approach to System Theory to address the problem at hand? In our opinion, one of the advantages is that a higher level of generality can be attained. For example, no input-output partition of the external variables is assumed, so that models of phenomena that do not allow a natural cause-effect relationship can be balanced, as can the more classical, input-output ones. Another reason is that no *a priori* choice of a state space representation is necessary, and a balanced state space representation is computed directly from the equations describing the model. This possibility makes the approach described here suitable in the common situation in which the behavior of a complex model obtained from first principles (and therefore consisting of a number of higher order equations, possibly with algebraic constraints), is to be simulated by a simple state space model.

## 2 Linear, time-invariant differential systems

In this section we introduce the notions of the behavioral approach to system theory more relevant to the problem at hand. The reader is referred to [11, 12] for more details.

For simplicity of exposition, in this note we consider systems with two external variables, which we will denote with $w_1$ and $w_2$. We will assume that these two quantities are related to each other via a linear, time-invariant, differential equation. Therefore, one way of describing the *behavior* $\mathcal{B}$ of the system, that is, the set of trajectories satisfying the laws of the system, is the *kernel representation*

$$p(\frac{d}{dt})w_1 = d(\frac{d}{dt})w_2, \tag{1}$$

where $p(\frac{d}{dt})$ and $d(\frac{d}{dt})$ are polynomial differential operators associated with the polynomials $p(\xi)$ and $d(\xi)$. For the moment, let us gloss over the issue of which function class $w_1$ and $w_2$ belong to, and let us introduce another representation for the behavior of the system.

If we assume that $p(\xi)$ and $d(\xi)$ are coprime, an alternative representation of the behavior (1) is the observable *image representation*

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} d(\frac{d}{dt}) \\ p(\frac{d}{dt}) \end{pmatrix} \ell \tag{2}$$

where $\ell$ is a *latent variable.* It can be shown that the set of trajectories $w = (w_1, w_2)$ such that (1) holds, coincides with the *external behavior* of (2), that is, with the set of trajectories $w$ for which there exists an $\ell$ such that (2) holds.

Of course the problem arises to what class of functions $w_1$, $w_2$ and $\ell$ of (1) and (2) belong. In the following, we will assume that they are $\mathcal{L}_2^{loc}(\mathbb{R},\mathbb{R})$ functions, with equality in (1) and (2) interpreted in the sense of distributions. It has been shown in [10] that every system in the form (1) with $w_1$, $w_2$ in $\mathcal{L}_2^{loc}(\mathbb{R},\mathbb{R})$ corresponds to a pair of coprime polynomials $p(\xi)$ and $q(\xi)$, and therefore in the following we deal without loss of generality with systems in the form (2).

A *state variable* is a special kind of latent variable satisfying the *axiom of state* (see [8]). It has been shown in [8] that a state variable can be induced by a *state map*, that is, by a polynomial differential operator that, acting on the variables of a system as $x = X(\frac{d}{dt})\begin{pmatrix} w \\ \ell \end{pmatrix}$, induces a new variable $x$ that satisfies the axiom of state. A state map is called *minimal* if the corresponding state variable $x$ is minimal, in the sense of having the least possible number of components. It can be shown that, starting from knowledge of a state map and of a representation in kernel or image form, one can compute quite directly a representation of the system consisting of equations of first order in $x$ and of zeroth order in $w$.

The following result characterizes the state maps acting on the latent variable. We denote with $n := max(deg\ p(\xi), deg\ d(\xi))$.

**Proposition 2.1** *Let a system be represented as in (2). A polynomial differential operator $X(\frac{d}{dt})$ acting on the latent variable $\ell$ is a state map if and only if the rows of the polynomial vector $X(\xi)$ span $\mathbb{R}_{n-1}[\xi]$, the vector space (over $\mathbb{R}$) of polynomials of degree not larger than $n-1$. $X(\frac{d}{dt})$ is a minimal state map if and only if the rows of $X(\xi)$ form a basis of $\mathbb{R}_{n-1}[\xi]$.*

It is easy to verify that $X(\frac{d}{dt})$ is a minimal state map if and only if $X(\xi) = T col(\xi^j)_{j=0,\dots,n-1}$, for some nonsingular matrix $T$.

## 3 Quadratic differential forms

In the context of many control and system theory problems, for example, in linear quadratic optimal control and Lyapunov stability theory, the analysis of the behavior of a quadratic functional of the system variables is performed (by minimization of the integral of a quadratic expression in the external variables of the system, in the case of optimal control; in the determination of a quadratic functional of the system variables, satisfying certain conditions in the case of Lyapunov theory). It has been shown in [9] that such quadratic functionals can be effectively represented by quadratic differential forms (QDF's). In this section we introduce some notation and some basic facts about QDF's; more details can be found in [9].

For the purposes of this paper, we will deal with differential functionals acting on two-dimensional system trajectories.

Let $\phi \in \mathbb{R}^{2\times 2}[\zeta,\eta]$ be a real symmetric polynomial matrix in the indeterminates $\zeta$ and $\eta$; that is, $\phi(\zeta,\eta) = \sum_{k,j=1}^{N} \phi_{k,j}\zeta^k\eta^j$, and $\phi(\zeta,\eta) = \phi(\eta,\zeta)^T$. We associate to $\phi$ a *bilinear differential functional* (BDF) $L_\phi$:

$$\begin{aligned} L_\phi \quad &: \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \times \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \\ L_\phi(v,w) \quad &:= \sum_{k,j} (\frac{d^k v}{dt^k})^T \phi_{k,j} (\frac{d^j w}{dt^j}). \end{aligned} \tag{3}$$

Note that this definition is given for infinitely differentiable trajectories $v$ and $w$. It can however be shown that under suitable conditions on $\phi(\zeta,\eta)$, the signals we deal with in the context of representations (2) exhibit enough smoothness for (3) to be a well defined function.

Of course, besides the BDF $L_\phi$, we can associate a QDF with the polynomial $\phi(\zeta,\eta)$:

$$\begin{aligned} Q_\phi &: \mathcal{C}^\infty(\mathbb{R},\mathbb{R})\times\mathcal{C}^\infty(\mathbb{R},\mathbb{R})\to\mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \\ Q_\phi(w) &:= \sum_{k,j}(\frac{d^k w}{dt^k})^T\phi_{k,j}(\frac{d^j w}{dt^j}) = L_\phi(w,w). \quad (4)\end{aligned}$$

Given a system described in image form (2) and a QDF $Q_\phi$, it is often convenient for technical reasons to consider the QDF *induced* by $Q_\phi$ on the latent variable $\ell$. This QDF is associated to the polynomial $\phi'(\zeta,\eta)$ defined as:

$$\phi'(\zeta,\eta) = \begin{pmatrix} d(\zeta) & p(\zeta)\end{pmatrix}\phi(\zeta,\eta)\begin{pmatrix} d(\eta)\\ p(\eta)\end{pmatrix}$$

and it satisfies, for all $w$ and $\ell$ for which (2) holds, $Q_\phi(w) = Q_{\phi'}(\ell)$. Note that $\phi'(\zeta,\eta)$ is a two variable polynomial. If it has degree $n$, we will say that $L_\phi$ (or $Q_\phi$) has degree $n$.

Let us now illustrate these concepts with an example. Assume that a representation (2) is given, and consider the squared $\mathcal{L}_2$-norm of the external variable $w$:

$$||w||_2^2 = \int_{-\infty}^{\infty}(w_1(t)^2 + w_2(t)^2)dt.$$

This norm can be effectively represented by the integral of the QDF $Q_\phi$ associated with

$$\phi(\zeta,\eta) = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$$

in the sense that

$$||w||_2^2 = \int_{-\infty}^{\infty} Q_\phi(w)(t)dt.$$

Note that this norm is induced also by the QDF $Q_{\phi'}$ acting on the latent variable $\ell$ and associated with the polynomial

$$\phi'(\zeta,\eta) = p(\zeta)p(\eta) + d(\zeta)d(\eta),$$

in the sense that

$$||w||_2^2 = \int_{-\infty}^{\infty} Q_{\phi'}(\ell)(t)dt.$$

In the sequel of the paper, the concept of positivity of a QDF will play an important role. We introduce this notion with reference to QDF's acting on the latent variable of a system represented as in (2). We will call $Q_\phi$ positive if $Q_\phi(\ell) > 0\ \forall\ \ell \in \mathcal{L}_2^{loc}(\mathbb{R},\mathbb{R})$. The question whether a quadratic differential form is positive can be effectively answered as follows (see [9]): $Q_\phi(\ell) \geq 0$ if and only if $\phi(\zeta,\eta) = r^T(\eta)r(\eta)$, where $r(\lambda) \neq 0\ \forall\ \lambda \in \mathbb{C}$. Note that this implies that $\phi(-i\omega, i\omega) > 0\ \forall\ \omega$.

## 4 The continuation and the antecedent map

The previous two sections introduced the basic definitions and tools that we need to approach the problem of balancing. In this section we use QDF's to measure the importance of a trajectory, via the introduction of a Hilbert space structure on the past and on the future of the behavior. We also introduce and study an operator which measures, in a way made precise in the following, the minimal effect that a system trajectory restricted to the past time axis $(-\infty,0)$ will have on the future behavior of the system.

Assume that a system (2) is given. Its external behavior $\mathcal{B}$ is the direct sum of the *past behavior* and of the *future behavior*, defined as

$$\begin{aligned}\mathcal{B}_- &:= \{w_- \mid w_- = w_{|(-\infty,0)},\ w\in\mathcal{B}\}\\ \mathcal{B}_+ &:= \{w_+ \mid w_+ = w_{|[0,\infty)},\ w\in\mathcal{B}\}.\end{aligned}$$

We will call $w_-$ the *past* of $w$, and $w_+$ the *future* of $w$. Observe that $w = w_- \wedge w_+$, with $\wedge$ denoting the *concatenation at zero*: $(f_1\wedge f_2)(t)$ equals $f_1(t)$ if $t<0$, and $f_2(t)$ if $t \leq 0$.

We now equip $\mathcal{B}_-$ and $\mathcal{B}_+$ with the structure of a Hilbert space. Let $L_{\phi_-}$ be a BDF acting on past trajectories and induced by a symmetric matrix $\phi_-$, such that the associated QDF $Q_{\phi_-}$ is positive. Define the inner product $<\cdot,\cdot>_-$ on $\mathcal{B}_-$ as

$$< w_-, w'_- >_- := \int_{-\infty}^{0} L_{\phi_-}(w_-, w'_-)(t)dt.$$

Let now $L_{\phi_+}$ be a BDF acting on future trajectories and induced by a symmetric matrix $\phi_+$, such that the associated QDF $Q_{\phi_+}$ is positive; define the inner product $<\cdot,\cdot>_+$ on $\mathcal{B}_+$ as

$$< w_+, w'_+ >_+ := \int_{0}^{\infty} L_{\phi_+}(w_+, w'_+)(t)dt.$$

We will assume that $\mathcal{B}_-$ and $\mathcal{B}_+$ are Hilbert spaces, with the norms induced by the given inner products. This implies that they are closed subspaces of $\mathcal{B}$ and that the inner products are positive.

Given a past trajectory $w_-$, any $w_+$ such that $w_-\wedge w_+ \in \mathcal{B}$ will be called a *continuation* of $w_-$. Given a future trajectory $w_+$, any $w_-$ such that $w_-\wedge w_+\in\mathcal{B}$ will be called an *antecedent* of $w_+$.

Of course, given a past trajectory $w_-$, in general there is a number of compatible continuations; intuitively, these continuations are future trajectories emanating from the same state to which $w_-$ led the system to. However, there exists only one continuation of $w_-$ with minimal squared norm. To see this, observe that since $p(\xi)$ and $d(\xi)$ of (2) are coprime, $w_-$ and $w_+$ correspond to uniquely determined past and future latent variable trajectories $\ell_-$ and $\ell_+$ (see [12]). Consider now the QDF $Q_{\phi'_+}$ acting on $\ell$ and induced by $Q_{\phi_+}$ (see section 3). For the sake of exposition, we will assume that $Q_{\phi'_+}$ has degree $n$. Since $Q_{\phi'_+}$ is nonnegative, there exists a Hurwitz polynomial $r_+(\xi)$ of degree $n$ such that $\phi'_+(\xi,-\xi) = r_+(\xi)r_+^*(\xi)$. Here $r_+^*(\xi) := r_+(-\xi)$. Define $(Ker\ r_+(\frac{d}{dt}))_+$ as

$$\{\ell'_+ = \ell'_{|[0,\infty)} \mid \ell' \in \mathcal{L}_2^{loc},\ (r_+(\frac{d}{dt})\ell')(t) = 0\ t \geq 0\},$$

and the *minimum energy system* (with time set $\mathbb{R}^+$) associated with $Q_{\phi_+}$, described by the equations

$$\begin{aligned} w_+ &= \begin{pmatrix} d(\frac{d}{dt}) \\ p(\frac{d}{dt}) \end{pmatrix} \ell_+ \\ 0 &= r_+(\frac{d}{dt})\ell_+ \end{aligned} \tag{5}$$

where the latent and the external variable have been subscripted with $+$ to emphasize that (5) is considered to hold for $t \geq 0$. The external behavior of (5) is a $n$-dimensional subspace of $\mathcal{B}_+$ consisting of linear combinations of Bohl functions whose characteristic exponents are the roots of $r_+(\xi)$. We will denote it with $\mathcal{B}_+^{opt}$. It can be shown that $\mathcal{B}_+^{opt}$ consists of the continuations of minimum squared norm. Since $\mathcal{B}_+^{opt}$ is closed, observe that $\mathcal{B}_+ = \mathcal{B}_+^{opt} \oplus (\mathcal{B}_+^{opt})^\perp$, with orthogonality defined in terms of the inner product $<\cdot,\cdot>_+$.

Analogously, for every $w_+ \in \mathcal{B}_+$ there exists a unique antecedent of minimum squared norm. Consider $Q_{\phi'_-}$, the quadratic differential form acting on $\ell$ induced by $Q_{\phi_-}$. For the sake of exposition we will assume that $Q_{\phi_-}$ has degree $n$. Let $\phi'_-(\xi,-\xi) = r_-(\xi)r_-^*(\xi)$, with $r_-(\xi)$ anti-Hurwitz. Denote with $(Ker\ r_-(\frac{d}{dt}))_-$ the set

$$\{\ell'_- = \ell'_{|(-\infty,0)} \mid \ell' \in \mathcal{L}_2^{loc},\ (r_-(\frac{d}{dt})\ell')(t) = 0\ t < 0\}.$$

It can be shown that the past behavior $\mathcal{B}_-$ is the direct sum of the finite dimensional external behavior of the minimum energy system (with time set $\mathbb{R}^-$)

$$\begin{aligned} w_- &= \begin{pmatrix} d(\frac{d}{dt}) \\ p(\frac{d}{dt}) \end{pmatrix} \ell_- \\ 0 &= r_-(\frac{d}{dt})\ell_-, \end{aligned} \tag{6}$$

which we will denote with $\mathcal{B}_-^{opt}$, and of the orthogonal subspace $(\mathcal{B}_-^{opt})^\perp$, with orthogonality defined in terms of the inner product $<\cdot,\cdot>_-$. $\mathcal{B}_-^{opt}$ is a $n$-dimensional subspace of $\mathcal{B}_-$ and consists of the antecedents of minimum squared norm.

Consider now the map $\Gamma_- : \mathcal{B}_- \to \mathcal{B}_+$ that associates to $w_-$ its future continuation of minimal squared norm, and the map $\Gamma_+ : \mathcal{B}_+ \to \mathcal{B}_-$ that maps $w_+$ into its antecedent of minimal squared norm. Formally,

$$\begin{aligned} \Gamma_-(w_-) &:= arg\ min\ \{\|w_+\|_+ \mid w_- \wedge w_+ \in \mathcal{B}\} \\ \Gamma_+(w_+) &:= arg\ min\ \{\|w_-\|_- \mid w_- \wedge w_+ \in \mathcal{B}\} \end{aligned}$$

We will call $\Gamma_-$ the *continuation map* and $\Gamma_+$ the *antecedent map.* It can be shown that $\Gamma_-$ and $\Gamma_+$ are well-defined, linear, continuous, and bounded. Moreover, $Im\ \Gamma_- = \mathcal{B}_+^{opt}$, $Im\ \Gamma_+ = \mathcal{B}_-^{opt}$, $Ker\ \Gamma_- = (\mathcal{B}_-^{opt})^\perp$, and $Ker\ \Gamma_+ = (\mathcal{B}_+^{opt})^\perp$. $\Gamma_-$ and $\Gamma_+$ are also compact, and it can be shown that they admit a Schmidt expansion

$$\begin{aligned} \Gamma_-(\cdot) &= \sum_{i=1}^{n} \mu_i < \cdot, w_i^- >_- w_i^+ \\ \Gamma_+(\cdot) &= \sum_{i=1}^{n} \frac{1}{\mu_i} < \cdot, w_i^+ >_- w_i^-. \end{aligned}$$

As we will show in the next section, the Schmidt expansion of $\Gamma_-$ is instrumental in the determination of a balanced state space representation. Let us therefore digress a bit and show how to compute the Schmidt pairs $(w_i^-, w_i^+)$ of $\Gamma_-$, $\Gamma_+$. Introduce the QDF's $Q_{\psi_+}$ and $Q_{\psi_-}$ acting on the latent variable and induced by the polynomials

$$\begin{aligned} \psi_+(\zeta,\eta) &:= \frac{r_+(\zeta)r_+^*(\eta) - \phi'_+(\zeta,\eta)}{\zeta+\eta} \\ \psi_-(\zeta,\eta) &:= \frac{\phi'_-(\zeta,\eta) - r_-(\zeta)r_-^*(\eta)}{\zeta+\eta}. \end{aligned}$$

We will call $Q_{\psi_+}$, $Q_{\psi_-}$ the *future* and the *past Gramian* associated with the QDF's $Q_{\phi'_+}$, $Q_{\phi'_-}$, respectively. It can be shown that, for trajectories satisfying (5) or (6), the value of $Q_{\psi_+}$ and of $Q_{\psi_-}$ at $t = 0$ coincides with the past, respectively, the future norm of the external signals of the system (analogously to the classical case, in which the controllability and observability Gramians measure the $\mathcal{L}_2$-norm of the inputs and the outputs). This allows us to state the following result:

**Proposition 4.1** *The Schmidt pairs $(w_i^-, w_i^+)$ of $\Gamma_-$ correspond to the trajectories $\ell_-$, $\ell_+$ of $(Ker\ r_-(\frac{d}{dt}))_-$, $(Ker\ r_+(\frac{d}{dt}))_+$, respectively, such that*

$$\begin{aligned} \mu_i^2 &:= \max\ Q_{\psi_+}(\ell_+)(0) \\ Q_{\psi_-}(\ell_-)(0) &= 1 \\ Q_{\psi_-}(\ell_-,\ell'_-)(0) &= 0 \\ \forall\ \ell'_-\ s.t.\quad w^- &\in V_{i-1}, \end{aligned} \tag{7}$$

*where the maximum is taken over all $\ell_+$ such that $\ell_- \wedge \ell_+$ is $n-1$ times continuously differentiable, and $V_i$ is defined as: $V_0 := \{0\}$, $V_i := span\ \{w_1^-, \ldots, w_i^-\}$, $i \geq 1$.*

The problem of finding $\ell_-$, $\ell_+$ that solve (7) can be shown to be equivalent to a generalized eigenvalue equation. We will not go into the details here.

## 5 Balanced state maps

We now define minimal balanced state maps.

**Definition 5.1** *A minimal state map $X(\frac{d}{dt})$ acting on the latent variable $\ell$ of the system (2) is said to be* balanced *if*

$$(X(\frac{d}{dt})\sqrt{\mu_i}\ell_i^+)(0) = \epsilon_i e_i \quad i = 1, \ldots, n$$

*where $e_i$ is the i-th vector of the canonical basis of $\mathbb{R}^n$, $\ell_i^+$ is the latent variable trajectory corresponding to the Schmidt vector $w_i^+$ and the singular value $\mu_i$ of $\Gamma_-$, and $\epsilon_i = \pm 1$.*

The rationale behind this definition is that by mapping the trajectories $w_i^- \frac{1}{\sqrt{\mu_i}} \wedge \sqrt{\mu_i} w_i^+$ at time $t = 0$ into the canonical basis, a choice of the state space is made so that states that correspond to a low energy of the past external trajectory and to a high energy of the future external trajectory are given more relevance. In this sense, the notion of balanced state map is a direct generalization of the concept of balanced state space basis. Note also that the idea of using a basis of suitably normalized Schmidt vectors to obtain a balanced realization is at the core of the results of [2, 3], where in place of $\Gamma_-$, the Hankel operator associated with the transfer function of the system is used.

A characterization of minimal balanced state maps can be given as follows. Consider the nonsingular $n \times n$ matrix $L\Sigma$ defined by

$$\begin{aligned} L &:= (\ell_{1+}^0 \ \cdots \ \ell_{n+}^0) \\ \Sigma &:= diag(\sqrt{\mu_i})_{i=1,\ldots,n}, \end{aligned} \tag{8}$$

where $\ell_{i+}^0 := col(\ell_{i+}^{(j)})_{j=0,\ldots,n-1}$ is the initial conditions vector of the trajectory $\ell_i^+$. The following result holds:

**Proposition 5.1** *$X(\frac{d}{dt})$ is a minimal balanced state map acting on $\ell$ for the system (2) if and only if $X(\xi) = S\Sigma^{-1}L^{-1}col(\xi^j)_{j=0,\ldots,n-1}$, where $S$ is a signature matrix.*

In the classical state space setting, balancing consists in the determination of a basis for the state space, such that the Gramians have a diagonal representation. The following proposition relates that point of view with the notion of balanced state map and of past and future Gramian put forward in this note.

**Proposition 5.2** *Let the system equations (2) and the future and past Gramians $Q_{\psi_+}$, $Q_{\psi_-}$ be given. If $X(\frac{d}{dt})$ is a balanced state map, then*

$$\begin{aligned} \psi_-(\zeta,\eta) &= X^T(\zeta)\Sigma^{-2}X(\eta) \\ \psi_+(\zeta,\eta) &= X^T(\zeta)\Sigma^{2}X(\eta) \end{aligned}$$

*where $\Sigma := diag(\sqrt{\mu_i})_{i=1,\ldots,n}$.*

The notion of balanced state map is instrumental for computing a balanced state space representation, from which a reduced order model is readily obtained. This is the subject of the next section.

## 6 Model reduction

The notion of balancing has been introduced in System Theory as a tool to obtain lower order state space models of dynamical systems. The idea underlying the procedure of model reduction is that once a balanced state space representation has been obtained, one can eliminate from the equations the state variables which contribute less to the external behavior of the system, obtaining a lower order model which behaves approximately as the original one.

We now show how to compute a reduced order state model for a system described by (1). The procedure starts by computing a balanced state map, from which balanced state space equations are readily obtained. To obtain a balanced state map, one has first to solve the $n$ problems (7) and to compute the latent trajectories $\ell_i^-$, $\ell_i^+$ corresponding to the Schmidt pairs $(w_i^-, w_i^+)$ of $\Gamma_-$. Let $L$ and $\Sigma$ be given as in (8), and let $X_b(\xi) := \Sigma^{-1}L^{-1}col(\xi^j)_{j=0,\ldots,n-1}$. As stated in proposition 5.1, $X_b(\frac{d}{dt})$ is a balanced state map.

Assume without loss of generality that $n = deg\ d(\xi)$.

There exists a polynomial $u(\xi)$ of degree $n$ and constant matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, $D \in \mathbb{R}$ such that

$$\begin{aligned} \xi X_b(\xi) &= AX_b(\xi) + Bu(\xi) \\ p(\xi) &= CX_b(\xi) + Du(\xi). \end{aligned} \tag{9}$$

To see this, consider that if $X(\xi) = col(\xi^j)_{j=0,\ldots,n-1}$, by choosing $u(\xi) = d(\xi)$, a realization of (1) in canonical controller form yields matrices $A_c$, $B_c$, $C_c$, $D_c$ such that

$$\begin{aligned} \xi X(\xi) &= A_cX(\xi) + B_cu(\xi) \\ p(\xi) &= C_cX(\xi) + D_cu(\xi). \end{aligned} \tag{10}$$

Since $X_b(\xi) = TX(\xi)$ with $T = \Sigma^{-1}L^{-1}$, (9) holds true with $A = TA_cT^{-1}$, $B = TB_c$, $C = C_cT^{-1}$, $D_c = D$.

In the case of Lyapunov balancing, it is a matter of straightforward verification to show that the realization $(A, B, C, D)$ corresponding to (9) is a balanced state space representation in the classical sense. In fact, it can be shown that the coordinate transformation represented by $\Sigma^{-1}L^{-1}$ diagonalizes the observability and the controllability Gramian.

Assume now that an approximate state space model of order $k < n$ is desired for the system (1). Partition $A$, $B$, $C$, of (9) as

$$\begin{aligned} A &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\ B &= \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \\ C &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \end{aligned}$$

where $A_{11}$ is $k \times k$, $B_1$ is $k \times 1$, $C$ is $1 \times k$. The desired model is described by the equations

$$\begin{aligned} \dot{x} &= A_{11}x + B_1 w_1 \\ w_2 &= C_1 x + D w_1. \end{aligned} \tag{11}$$

## 7 Conclusions

In this note we have illustrated a framework for model reduction by balancing for systems described by high-order differential equations. This point of view is particularly significant in that it allows the computation of a reduced order state space model of a complex system described by higher-order linear differential equations, without the need to compute an intermediate state space model. We did not go into the details in this note, but it can be shown that classical Lyapunov and Riccati balanced model reduction are special cases of the approach pursued here. Moreover, in these two cases many of the computations necessary to obtain a balanced state map can be done polynomially, much like what has been done in [2, 3].

Our research efforts are currently aimed in two directions. The first one is the possibility of obtaining the image representation corresponding to the reduced order state space model (11) directly from knowledge of a balanced state map and of the equations (2). The second, more ambitious one, is to study the connections between the notion of balanced state map with the problem of Hankel-norm approximation.

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