Reachability properties
of discrete-time positive switched systems

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Joint work with E. Fornasini
Workshop “Open and Interconnected Systems Modeling and Control”
Brugge, September 16-17, 2009
Outline

- Motivations

- Preliminaries

- Single-input discrete-time positive switched systems and reachability properties

- Monomial reachability of a class of single-input positive switched systems

- Reachability of a class of single-input positive switched system
Motivations

Consider a continuous-time linear compartmental system:

\[
\dot{x}(t) = Ax(t) + Bu(t),
\]

with \( A \) a Metzler matrix and \( B \) a positive matrix. Multi-rate sampling of the above model, with sampling interval lengths \( \{T_1, T_2, \ldots, T_p\} \), brings to a discrete-time model of the following form

\[
x(t + 1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t),
\]

where \( \sigma \) is a “switching sequence”, defined on \( \mathbb{Z}_+ \) and taking values in \( \{1, 2, \ldots, p\} \). At each time \( t \) the switching system takes the form

\[
x(t + 1) = A_i x(t) + B_i u(t),
\]

where \( i = \sigma(t) \in \{1, 2, \ldots, p\} \), while \( A_i = e^{AT_i} \) and \( B_i = \int_0^{T_i} e^{A\tau} Bd\tau \) are positive matrices.
Preliminaries

- $e_i = [0 \ldots 0 1 0 \ldots 0]^T$ is the $i$th vector of the canonical basis in $\mathbb{R}^n$;

- $\mathbb{R}_+$ is the semiring of nonnegative real numbers, and $\mathbb{R}_+^n$ the positive orthant;

- a matrix $A$ (a vector $v$) is nonnegative if all its entries are in $\mathbb{R}_+$ and positive if nonnegative and nonzero;

- a vector $v$ is a monomial vector if $v = \alpha e_i$ for some index $i$ and some positive $\alpha$;

- a monomial matrix is a nonsingular square nonnegative matrix whose columns are monomial vectors. A monomial matrix whose nonzero entries are unitary is a permutation matrix.
Single-input discrete-time positive switched systems and reachability properties

A single-input discrete-time positive switched system is described as

\[ x(t + 1) = A_{\sigma(t)}x(t) + b_{\sigma(t)}u(t), \]  \( (1) \)

where

- \( x \) is the \( n \)-dimensional \text{state variable};
- \( u \) the \text{scalar input};
- \( \sigma \) is a (piece-wise constant and right continuous) switching sequence, taking values in the finite set \( P = \{1, 2, \ldots, p\} \).

For each \( i \in P \), the pair \( (A_i, b_i) \) represents a discrete-time positive system: \( A_i \) is an \( n \times n \) \text{nonnegative} matrix and \( b_i \) is an \( n \)-dimensional \text{nonnegative} vector.
Definition 1 A state $x_f \in \mathbb{R}^n_+$ is said to be reachable at time $k \in \mathbb{N}$
if there exist a switching sequence $\sigma : \mathbb{Z}_+ \rightarrow \mathcal{P}$
and an input sequence $u : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$
that lead the state trajectory from $x(0) = 0$
to $x(k) = x_f$.

A positive switched system is monomially reachable if every monomial
vector $x_f \in \mathbb{R}^n_+$ is reachable at some time $k$.

A positive switched system is reachable if every state $x_f \in \mathbb{R}^n_+$
is reachable at some time $k$.

It follows immediately from Definition 1 that reachability implies
monomial reachability. However, the converse is not true, differently from
what happens with standard positive systems.
Monomial reachability of a class of single-input positive switched systems

In this talk, we focus on the class of single-input discrete-time positive switched systems (dPSS) described by the state equation

\[ x(t + 1) = Ax(t) + b_{\sigma(t)}u(t). \]  

This means that the \( p = |\mathcal{P}| \) subsystems among which the system switches share the same nonnegative system matrix \( A \), and differ only in the input-to-state matrices, \( b_i \in \mathbb{R}_+^n, i \in \mathcal{P} \).
To explore reachability properties for system (2), consider the expression of the state at any time instant \( k \in \mathbb{N} \), starting from the initial condition \( x(0) = 0 \), under the effect of the input sequence \( u(0), u(1), \ldots, u(k - 1) \) and of the switching sequence \( \sigma(0), \sigma(1), \ldots, \sigma(k - 1) \):

\[
x(k) = A^{k-1}b_{\sigma(0)}u(0) + A^{k-2}b_{\sigma(1)}u(1) + \ldots + Ab_{\sigma(k-2)}u(k - 2) + b_{\sigma(k-1)}u(k - 1),
\]

(3)

where \( \sigma(t) \in \mathcal{P} \) for every \( t \in \{0, 1, \ldots, k - 1\} \). If we introduce the reachability matrix associated with the switching sequence \( \sigma \) of length \( k \):

\[
\mathcal{R}_k(\sigma) = \begin{bmatrix}
A^{k-1}b_{\sigma(0)} & A^{k-2}b_{\sigma(1)} & \ldots & Ab_{\sigma(k-2)} & b_{\sigma(k-1)}
\end{bmatrix},
\]

(3) can be rewritten as

\[
x(k) = \mathcal{R}_k(\sigma) \begin{bmatrix}
u(0) \\
\vdots \\
u(k - 1)
\end{bmatrix}.
\]
As the input samples $u(0), \ldots, u(k - 1)$ are nonnegative, $x(k) \in \text{Cone}(\mathcal{R}_k(\sigma))$.

So, a positive state $x_f$ is reachable if and only if there exists a switching sequence $\sigma$ such that $x_f \in \text{Cone}(\mathcal{R}_{|\sigma|}(\sigma))$. 
Proposition 1 Given a dPSS (2), the following are equivalent:

i) the switched system (2) is monomially reachable;

ii) the non-switched positive system \( x(t + 1) = Ax(t) + Bu(t), \) with

\[
B := \begin{bmatrix} b_1 & b_2 & \ldots & b_p \end{bmatrix},
\]

is reachable, namely its reachability matrix

\[
\mathcal{R}_n(A, B) := \begin{bmatrix} A^{n-1}B & \ldots & AB & B \end{bmatrix},
\]

contains an \( n \times n \) monomial submatrix.
Reachability of a class of single-input positive switched systems

We now address the broader problem of reachability for the dPSS (2) by assuming $\mathcal{P} = \langle 2 \rangle$ (switching between two subsystems sharing the same system matrix). The extension to the case when $|\mathcal{P}| \geq 2$ is more articulate, and has been derived in a recently submitted paper.

A preliminary technical result.

Lemma 1 If the dPSS (2), commuting between $p = 2$ subsystems, is reachable, then

1. the $n \times (n + 2)$ positive matrix $[ A \ b_1 \ b_2 ]$ includes an $n \times n$ monomial matrix,

2. $A$ is not zero, and

3. at least one of the vectors $b_i, i \in \mathcal{P} = \{1, 2\}$, is a monomial vector.
As a consequence, in the following we will assume w.l.o.g. that:

- $b_1$ is a monomial vector and, specifically, $b_1 = e_1$;
- $A$ is not zero;
- $b_1$ and $b_2$ are linearly independent (if not, reachability of the dPSS (2) would reduce to the reachability of a single-input positive system).
• SYSTEM DIMENSION $n = 2$.

Proposition 2 Consider a dPSS (2), with $A \in \mathbb{R}^{2 \times 2}_+, A \neq 0, b_1 = e_1$, and $b_2 \in \mathbb{R}^2_+$ linearly independent of $b_1$.

The 2-dimensional system (2) is reachable if and only if either

i) $[b_1 \quad b_2]$ is a $2 \times 2$ monomial matrix, or

ii) $A = \begin{bmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, and $a_{21} = 0$ implies $a_{12} = 0$. 
2 cases: (1) only $b_1$ is a monomial vector, addressed in Theorem 1, 
(2) both $b_1$ and $b_2$ are monomial vectors, dealt with in Theorem 2.

**Theorem 1** Consider a dPSS (2), with $\mathcal{P} = \langle 2 \rangle$, $A \in \mathbb{R}^{n \times n}_+$, $b_1, b_2 \in \mathbb{R}^n_+$, $b_1$ monomial and $b_2$ a nonzero non-monomial vector. Suppose $(A, b_1)$ is not reachable. The dPSS (2) is reachable if and only if there exist $r \in \mathbb{N}$, and a permutation matrix $P$, such that

$$P^T A P = \begin{bmatrix}
0 & \ldots & 0 & 0 \\
a_{2,1} & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & a_{r,r-1} & 0 \\
0_{(n-r) \times r} & a_{r+2,r+1} & \ldots & 0_{r \times (n-r)} \\
\vdots & \ddots & \vdots & \vdots \\
0_{(n-r) \times r} & \vdots & \ddots & \vdots \\
0 & \ldots & a_{n,n-1} & 0
\end{bmatrix},$$

where $a_{i+1,i} > 0$, and (possibly after a rescaling) $P^T b_1 = e_1$, while $P^T b_2 = e_{r+1} + \sum_{i=1}^{r} \alpha_i e_i$, and at least one of the $\alpha_i$’s is positive.
Theorem 2 Consider a dPSS (2), with $\mathcal{P} = \langle 2 \rangle$, $A \in \mathbb{R}^{n \times n}_+, b_1, b_2 \in \mathbb{R}^n_+$, $b_1$ and $b_2$ two linearly independent monomial vectors. Suppose that neither $(A, b_1)$ nor $(A, b_2)$ are reachable subsystems. The dPSS (2) is reachable if and only if there exist $r \in \mathbb{N}$, and a permutation matrix $P$, such that

$$P^TAP = \begin{bmatrix}
0 & \ldots & 0 & a_{1r} & 0 & \ldots & 0 & a_{1n} \\
0 & \ldots & 0 & a_{2r} & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & a_{r,r-1} & a_{rr} & 0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & a_{r+1,r} & 0 & \ldots & 0 & a_{r+1,n} \\
0 & \ldots & 0 & a_{r+2,r} & a_{r+2,r+1} & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & a_{nr} & 0 & \ldots & a_{n,n-1} & 0 
\end{bmatrix}$$

where $a_{i+1,i} > 0$ for $i \in \{1, 2, \ldots, r-1\} \cup \{r+1, r+2, \ldots, n-1\}$, $a_{r+1,n} > 0$, $a_{1n} \geq 0$, and (possibly after a rescaling) $P^Tb_1 = e_1$ and $P^Tb_2 = e_{r+1}$. 

Open and Interconnected Systems Modeling and Control, 2009
Conclusions

- In this talk we addressed the reachability problem for a class of single-input discrete-time positive switched systems.

- Necessary and sufficient conditions for monomial reachability to hold, as well as necessary and sufficient conditions for reachability, assuming that the system switches just between two subsystems have been provided.

- For this class of positive switched systems, reachability is a structural property: it depends only on the nonzero patterns of the matrices $A, b_1$ and $b_2$ involved.