# Unsupervised Self-Adaptive Auditory Attention Decoding: Supplementary Material

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In the supplementary material, we show convergence to a unique fixed point of the fixed-point iteration on the updating model (Equation (16) in the original paper). We hypothesize that under three reasonable conditions on the accuracies of the attended and unattended decoder, there exists a unique fixed point $p^*$ to which the fixed-point iteration $p_{i+1} = \phi(p_i)$ converges, starting from any (possibly random) decoder. In Section I, we first show that there always exists such a fixed point, while in Section II we check the uniqueness of and convergence to this fixed point under the hypothesized conditions.

## I. Existence

Consider the following fixed-point theorem, also known as Brouwer's fixed-point theorem [1]:

**Theorem 1 (Brouwer's fixed point theorem [1]).** *Any continuous self map of a nonempty compact convex subset of a Euclidean space has a fixed point.*

As the function $\phi(p_i)\colon [0,100]\% \to [0,100]\%$ in (16) is a continuous function that maps its domain onto itself and $[0,1]$ is a closed (thus, compact) convex subset of $\mathbb{R}$, Brouwer's fixed point theorem assures that there exists at least one fixed point.

## II. Uniqueness and convergence

We evaluate the model in (16) in a relevant range of the parameters $\mu_1, \mu_2$, and $\sigma$, obeying three reasonable conditions, to show the convergence to a unique fixed point.

### A. Three conditions for convergence

Consider the supervised subject-specific attended decoder $\hat{\mathbf{d}}_\text{a}$ with accuracy $p_\text{a}$ (on the attended labels) and supervised subject-specific unattended decoder $\hat{\mathbf{d}}_\text{u}$ with accuracy $p_\text{u}$ (on the unattended labels). We then a priori postulate the following three intuitive and reasonable conditions on the accuracies $p_\text{a}$ and $p_\text{u}$ (which will turn out to be satisfied for all subjects in both datasets):

- $p_\text{a} - p_\text{u} > 5\%$, i.e., the attended decoder needs to perform $5\%$ better (on the attended labels) than the unattended decoder (on the unattended labels). Given that the attended speech envelope is typically better represented in the EEG, we indeed expect a difference in performance between both decoders. Moreover, this condition can be linked to the expectation that the cross-correlation between the EEG and attended speech envelope is on average larger than with the unattended speech envelope, serving as a possible explanation for the self-leveraging effect (see Section IV-B in the original paper).
- $p_\text{u} < 85\%$, i.e., the *unattended* decoder may not perform better than $85\%$ (on the unattended labels). If the unattended decoder performs too well, then, again, the self-leveraging effect may not be present for the same reason as mentioned in the previous condition.
- $p_\text{a} > 100\% - p_\text{u}$, i.e., the attended decoder is better at predicting attended labels than the unattended decoder. This assures that the starting point of the model curve $\phi(0\%) = 100\% - p_\text{u}$ (e.g., see Figure 2 in the original paper) is below the end point $\phi(100\%) = p_\text{a}$.

In the following sections, we will use the model in (16) to show that there is convergence to a unique fixed point when these three conditions are satisfied. However, it is noted that these postulated conditions are conservative in the mathematical sense, i.e., they are 'sufficient' but *not* 'necessary' conditions. When they are not satisfied, there can still be convergence to a unique fixed point.

Moreover, the three conditions are also intuitive and very reasonable from a practical point of view, as they are satisfied for all subjects in both datasets; the minimum across all subjects of $p_\text{a} - p_\text{u} = 8.3\% > 5\%$, the maximum across all subjects of $p_\text{u} = 76.7\% < 85\%$, and the minimum across all subjects of $p_\text{a} + p_\text{u} = 124\% > 100\%$.

### B. Convergence to a unique fixed point

Consider the following fixed-point theorem that provides sufficient conditions for convergence to a unique fixed point of the fixed-point iteration $p_{i+1} = \phi(p_i)$ [2]:

**Theorem 2.** *Let $\phi$ be a continuous function on $[a,b]$, such that $\phi(p_i) \in [a,b], \forall\, p_i \in [a,b]$, and suppose that $\phi'$ exists $\forall\, p_i \in [a,b]$ and that a constant $0 < \alpha < 1$ exists such that:*

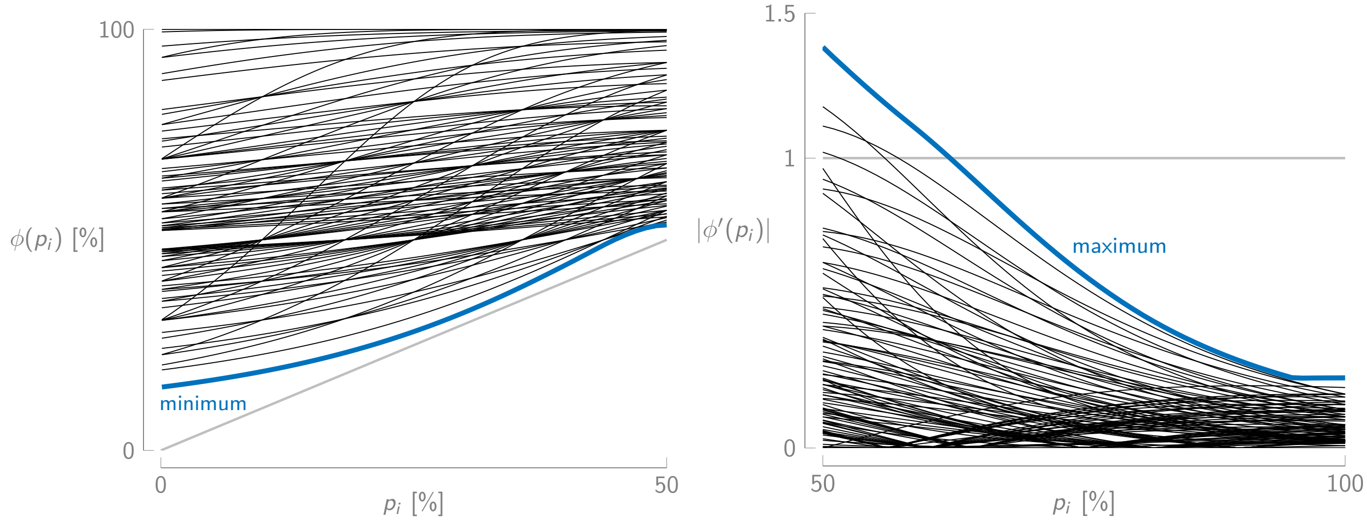
$$|\phi'(p_i)| \leq \alpha, \forall\, p_i \in [a,b],$$

*then there is exactly one fixed point $p^* \in [a,b]$ and the fixed-point iteration $p_{i+1} = \phi(p_i)$ will converge to this unique fixed point in $[a,b]$.*

We now evaluate the model $\phi(p_i)$ in (16) and its derivative $\phi'(p_i)$ to show convergence to a unique fixed point based on Theorem 2 for the case where the conditions in Section II-A are satisfied.

The derivative $\phi'(p_i)$ of the model in (16) can be computed by hand or by using any symbolic math software and is equal to:

$$\phi'(p_i) = \frac{p_i \sigma_z(p_i)^2 \mu_2 + (1-p_i)\sigma^2 \mu_z(p_i)}{\sqrt{2\pi} p_i^3 \sigma_z(p_i)^3} e^{-\frac{1}{2}\left(\frac{\mu_z(p_i)}{\sigma_z(p_i)}\right)^2}. \quad (1)$$




(a) (b)

Figure 1: (a) A subset of the evaluated $\phi(p_i)$ for $p_i \in [0, 50]\%$ and the minimum over all evaluated $(\mu_1, \mu_2, \sigma)$ that obey the conditions are all above the identity line, where $\phi(p_i) = p_i$, which shows that $\phi(p_i) > p_i, \forall\, p_i \in [0, 50]\%$. (b) A subset of the evaluated $|\phi'(p_i)|$ for $p_i \in [50, 100]\%$, together with the maximum over all evaluated $(\mu_1, \mu_2, \sigma)$ that obey the conditions.

To evaluate (16) and its derivative (1), we take 300 equidistant samples of $\mu_1 \in [-2, 2]$, 300 equidistant samples of $\mu_2 \in [-2, 2]$, and 100 equidistant samples of $\sigma \in ]0, 4]$. These intervals contain the complete range of parameters concerning the difference in correlation coefficients $R_1$ and $R_2$. From this parameter range, we select all combinations of $(\mu_1, \mu_2, \sigma)$ for which the three conditions of Section II-A are satisfied. The connection between $p_\text{a}$ and $p_\text{u}$ (as used in the three conditions) and the model parameters $(\mu_1, \mu_2, \sigma)$ is given by:

$$p_\text{a} = P(R_1 > 0) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma}\right)^2} dx \text{ and}$$

$$p_\text{u} = P(R_2 > 0) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma}\right)^2} dx,$$

using the assumptions in Section IV-A in the original paper. These connections can be derived from the updating model in Equation (16) from the original paper by setting $p_i = 100\%$, resp. $p_i = 0\%$, resulting in the decoder accuracy of the supervised attended, resp. unattended decoder.

Figure 1a now shows a subset of $\phi(p_i)$ for $p_i \in [0, 50]\%$, for all evaluated $(\mu_1, \mu_2, \sigma)$ that obey the three conditions, together with the minimum over all these $\phi(p_i)$. Similarly, Figure 1b shows a subset of $|\phi'(p_i)|$ for $p_i \in [50, 100]\%$, for all evaluated $(\mu_1, \mu_2, \sigma)$ that obey the three conditions, together with the maximum over all these $|\phi'(p_i)|$. Both results are required to show convergence to a unique fixed point using Theorem 2:

- **Result 1**: From Figure 1a, it can be seen that $\phi(p_i) > p_i, \forall\, p_i \in [0, 50]\%$. This implies that there is no fixed point within this interval and that the fixed-point iteration will always diverge to the $p_i \in [50, 100]\%$ interval. This is because $\forall\, p_i \in [0, 50]\% : p_{i+1} = \phi(p_i) > p_i$, i.e., the new accuracy in the fixed-point iteration is always larger than the previous one, such that, inevitably, at a certain iteration, $p_{i+1} > 50\%$. It thus suffices to show that there is convergence to a unique fixed point for $p_i \in [50, 100]\%$, which is shown in the next result.
- **Result 2**: From Figure 1b, there are two possible cases, which both individually can be shown to guarantee convergence to a unique fixed point:
  1) $|\phi'(p_i)| < 1, \forall\, p_i \in [50, 100]\%$. For all these cases, we then numerically confirmed that $\phi(p_i) \in [50, 100]\%, \forall\, p_i \in [50, 100]\%$ such that all conditions of Theorem 2 are fulfilled to show convergence to a unique point.
  2) $\exists\, x \in [50, 100]\% : \phi'(p_i) \geq 1, \forall\, p_i \in [50, x]\%$ and $|\phi'(p_i)| < 1, \forall p_i \in [x, 100]\%$. Since $\phi(50\%) > 50\%$ (see Result 1) and since the derivative is positive, it is guaranteed that $\phi(p_i) > p_i, \forall\, p_i \in [50, x]\%$, i.e., there is no fixed point and the fixed-point iteration diverges to the $p_i \in [x, 100]\%$ interval (using a similar reasoning as in Result 1). Furthermore, it can again be numerically checked that $\phi(p_i) \in [x, 100]\%, \forall\, p_i \in [x, 100]\%$ to show that there is a unique point to which there is convergence in this interval (see Theorem 2).

## References

[1] J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization: Theory and Examples*, 2nd ed. Springer-Verlag New York, 2006.
[2] Walter Gautschi, *Numerical Analysis*. Birkhäuser Basel, 2012.